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A Higher Order Method for Determining Nonisolated Solutions of a System of Nonlinear Equations*

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Abstract — Zusammenfassung

A Higher Order Method for Determining Nonisolated Solutions of a System of Nonlinear Equations. In this note, we obtain a method of order at least four to solve a singular system of nonlinear algebraic equations. This is achieved by enlarging the system to a higher dimensional one whose solution is isolated. For the larger system we use a method developed by B. Neta.

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Eine Methode höherer Ordnung zur Bestimmung nicht-isolierter Lösungen eines nichtlinearen Gleichungssystems. In dieser kurzen Mitteilung gewinnen wir eine Methode mindestens vierter Ordnung zur Lösung eines singulären Systems nichtlinearer algebraischer Gleichungen. Dies wird durch eine Vergrößerung des Gleichungssystems zu einem höherdimensionalen Gleichungssystem, dessen entsprechende Lösung isoliert ist, erreicht. Dessen Lösung kann durch eine von B. Neta entwickelte Methode bestimmt werden.

1. Introduction

Newton's method for determining nonisolated solutions of nonlinear algebraic equations was discussed by Rall [6], Cavanaugh [1], Ortega and Rheinboldt [5], Reddien [7, 8], Decker and Kelley [2, 3], and recently by Weber and Werner [9]. In [9], Weber and Werner suggested solving an auxiliary problem of higher dimension which has an isolated solution. For the auxiliary problem they suggest using an algorithm developed by Werner [10, 11] of order $1+\sqrt{2}$. In this note, we suggest solving the same auxiliary problem by a method of order at least four (Neta [4]).

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2. An Auxiliary Problem

Let $\mathfrak{X}$ be a Banach space and $F \in C^2(\mathfrak{X})$. Let $F(x^*)=0$ and $\dim \mathfrak{N}(F'(x^*))=1$ where $\mathfrak{N}(F'(x^*))$ is the null space of $F'(x^*)$. Let $\mathfrak{Y}$ be closed, $F'(x^*)\mathfrak{Y}=\mathfrak{Y}$ and

$$\mathfrak{X}=\mathfrak{N}\oplus\mathfrak{Y}. \tag{1}$$

Furthermore, let

$$F''(x^*)\mathfrak{N}\mathfrak{N}\cap\mathfrak{Y}=\{0\}, \tag{2}$$

$$\|F''(x^*)nx\|\geq c\,\|n\|\,\|x\|,\ c>0, \text{ for all } n\in\mathfrak{N},\ x\in\mathfrak{X}. \tag{3}$$

Let $\mathfrak{B}=\mathfrak{X}\times\mathfrak{X}\times R$ be equipped with the norm

$$\|b\|_{\mathfrak{B}}=\|x_1\|_{\mathfrak{X}}+\|x_2\|_{\mathfrak{X}}+|r| \tag{4}$$

where $b=(x_1,x_2,r)$, $x_1,x_2\in\mathfrak{X}$, $r\in R$. In the case 0 is an eigenvalue of algebraic multiplicity one, the higher dimensional system is

$$\boldsymbol{F}(b)=\boldsymbol{F}(x,y,\lambda)=\begin{pmatrix} F(x)+\lambda y\\ F'(x)y\\ a(y,y)-1\end{pmatrix}; \tag{5a}$$

for the case where 0 is an eigenvalue of geometric multiplicity one, but algebraic multiplicity greater than one, the mapping $\boldsymbol{F}$ is

$$\boldsymbol{F}(b)=\boldsymbol{F}(x,y,\lambda)=\begin{pmatrix} F'(x)^T F(x)+\lambda y\\ F'(x)y\\ a(y,y)-1\end{pmatrix}, \tag{5b}$$

where a is a continuous, symmetric, positive definite bilinear form on $\mathfrak{X}\times\mathfrak{X}$, and the superscript "$T$" in (5 b) denotes transpose. In either case, Weber and Werner [9] showed that the system

$$\boldsymbol{F}(b)=0 \tag{6}$$

has an isolated solution $b^*=(x^*,y^*,0)$, since we take the vector $y^*\in\mathfrak{N}$ to be such that $a(y^*,y^*)=1$.

3. Numerical Solution of the Auxiliary Problem

Since the solution $(x^*,y^*,0)$ of (6) is isolated, one could apply the following algorithm developed in [4] for computing solutions of nonlinear systems. This algorithm is of R-order at least four and requires three function evaluations and one evaluation of the Jacobian ($\boldsymbol{F}'$) per step.

(i) Given b_k, $\boldsymbol{F}(b_k)$, $\boldsymbol{F}'(b_k)$:

(ii) Solve

$$\boldsymbol{F}'(b_k)(w_k-b_k)=-\boldsymbol{F}(b_k) \tag{7}$$

for w_k.

(iii) Evaluate $\boldsymbol{F}(w_k)$ and test for convergence. Either terminate the computation or proceed to (iv).

(iv) Evaluate the entries of the diagonal matrix $\boldsymbol{D}$

$$\boldsymbol{D}_{ii} = \begin{cases} \dfrac{\boldsymbol{F}_i(b_k) - \boldsymbol{F}_i(w_k)}{\boldsymbol{F}_i(b_k) - 3\,\boldsymbol{F}_i(w_k)} & \text{if denominator } \neq 0 \\ 1 & \text{otherwise.} \end{cases} \tag{8}$$

(v) Solve

$$\boldsymbol{F}'(b_k)(z_k - w_k) = -\boldsymbol{D}\,\boldsymbol{F}(w_k) \tag{9}$$

for z_k.

(vi) Evaluate $\boldsymbol{F}(z_k)$ and test for convergence. Either terminate the computation or proceed to (vii).

(vii) Solve

$$\boldsymbol{F}'(b_k)(b_{k+1} - z_k) = -\boldsymbol{D}\,\boldsymbol{F}(z_k) \tag{10}$$

for b_{k+1}.

(viii) Evaluate $\boldsymbol{F}(b_{k+1})$ and test for convergence. Either terminate the computation or proceed to (ix).

(ix) Evaluate $\boldsymbol{F}'(b_{k+1})$, set the counter to $k+1$, and return to (ii).

It was shown in [4] that this algorithm is of R-order at least 4. Numerical experiments described in [4] show that one can save over 20% of the cost of solving a system of algebraic equations. The saving is greater when the dimension is higher or the number of iterations needed is larger.

Remark: Since the dimension of the system is more than doubled and since the saving is an increasing function of the dimension, it makes even more sense to use our algorithm.

Let us compare the number of multiplications required by one step of our algorithm with that of Werner's. Let n be the dimension of the original system. Let N_c be the number of multiplications required in calculating the entries of the Jacobian matrix, and N_F the number required for factorization of that matrix. Then,

$$N_c \sim 2\,n^2, \tag{11}$$

$$N_F \sim n^3. \tag{12}$$

The number of multiplications required in the step of back substitution is

$$N_s \sim 2\,n^2. \tag{13}$$

The number, N_D, of multiplications required in calculating the entries of the diagonal matrix $\boldsymbol{D}$ and multiplying by $\boldsymbol{F}$ is

$$N_D \sim 4\,n. \tag{14}$$

The total number of multiplications required by our algorithm is then

$$T_N = N_c + N_F + 3\,N_s + N_D$$

$$T_N = n(n^2 + 8\,n + 4). \tag{15}$$

The total number of multiplications required by Werner's algorithm is

$$T_w = N_c + N_F + 2\, N_s,$$
$$T_w = n^2 (n+6). \tag{16}$$

Let us now define the efficiency of an algorithm as follows:

$$e = \frac{p}{T}, \tag{17}$$

where p is the order and T is the total number of multiplications per step.

The efficiency of our algorithm is

$$e_N = \frac{4}{T_N} = \frac{4}{n(n^2 + 8\,n + 4)}. \tag{18}$$

The efficiency of Werner's algorithm is

$$e_w = \frac{1 + \sqrt{2}}{T_w} = \frac{1 + \sqrt{2}}{n^2 (n+6)}. \tag{19}$$

It can be shown easily that, for $n \geq 2$, our algorithm is more efficient than Werner's.

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